Semester-I
Vector Analysis

Vector Operators: Grad, Div and Curl

- the gradient of a scalar field,
- the divergence of a vector field, and
- the curl of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

[redacted] The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how to.

If $U(x, y, z)$ is a scalar field, ie a scalar function of position $\mathbf{r} = [x, y, z]$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\text{grad}U = \frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} .$$

It is usual to define the **vector operator**

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}$$

which is called “del” or “nabla”. Then

$$\text{grad}U \equiv \boldsymbol{\nabla} U$$

Note immediately that ∇U is a vector field!

Without thinking too carefully about it, we can see that the gradient tends to point in the direction of greatest change of the scalar field. Later we will be more precise.

♣ Worked examples of gradient evaluation

1. $U = x^2$

   Only $\partial/\partial x$ exists so

   $$\nabla U = 2x\hat{\imath} \ .$$

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2. $U = r^2$

   $r^2 = x^2 + y^2 + z^2$, so

   $$\begin{aligned} \nabla U &= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{\boldsymbol{k}} \\ &= 2\,\mathbf{r} \end{aligned}$$

   .

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3. $U = \mathbf{c}\cdot\mathbf{r}$, where $\mathbf{c}$ is constant.

   $$\nabla U = \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\boldsymbol{k}} = \mathbf{c} \ .$$

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4. $U = f(r)$

   $$U = f(x, y, z) = f(r(x, y, z))$$

   Now $\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}$, $\frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y}$, and $\frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z}$, so

   $$\nabla U = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} = \frac{df}{dr}\left[\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}}\right]$$

   But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z. Hence

   $$\nabla U = \frac{df}{dr}\left[\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}}\right] = \frac{df}{dr}\left[\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}}{r}\right] = \frac{df}{dr}\left[\frac{\mathbf{r}}{r}\right] \ .$$

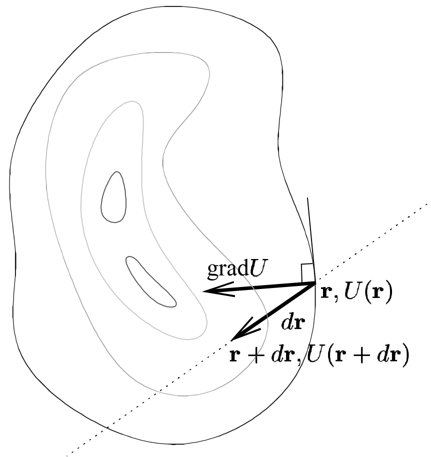

Figure 1: The directional derivative

The significance of grad

We have seen that

$$\nabla U = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}}$$

so if we move a small amount $d\mathbf{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{\boldsymbol{k}}$ the change in U is (see figure 1)

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \nabla U \cdot d\mathbf{r}\ .$$

Now divide by ds

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds}\ .$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.

So

- grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**. Note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that

- At any point P, grad$U$ points in the direction of greatest change of U at P, and has magnitude equal to the rate of change of U wrt distance in that direction.

Another nice property emerges if we think of a surface of constant U – that is the locus (x, y, z) for

$$U(x, y, z) = \text{constant}$$

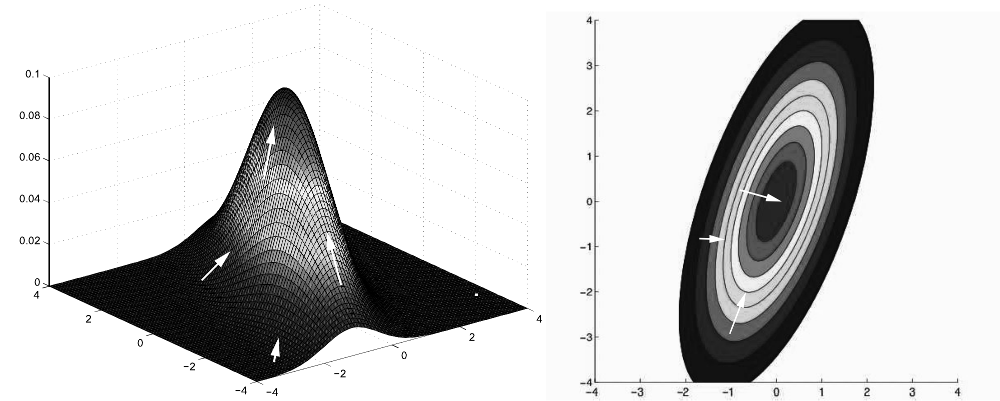

Figure 2:

If we move a tiny amount within the surface, that is in any tangential direction, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

$$\boldsymbol{\nabla} U \cdot \frac{d\mathbf{r}}{ds} \;=\; 0\,.$$

This can only be satisfied if

- grad$U$ is NORMAL to a surface of constant U.

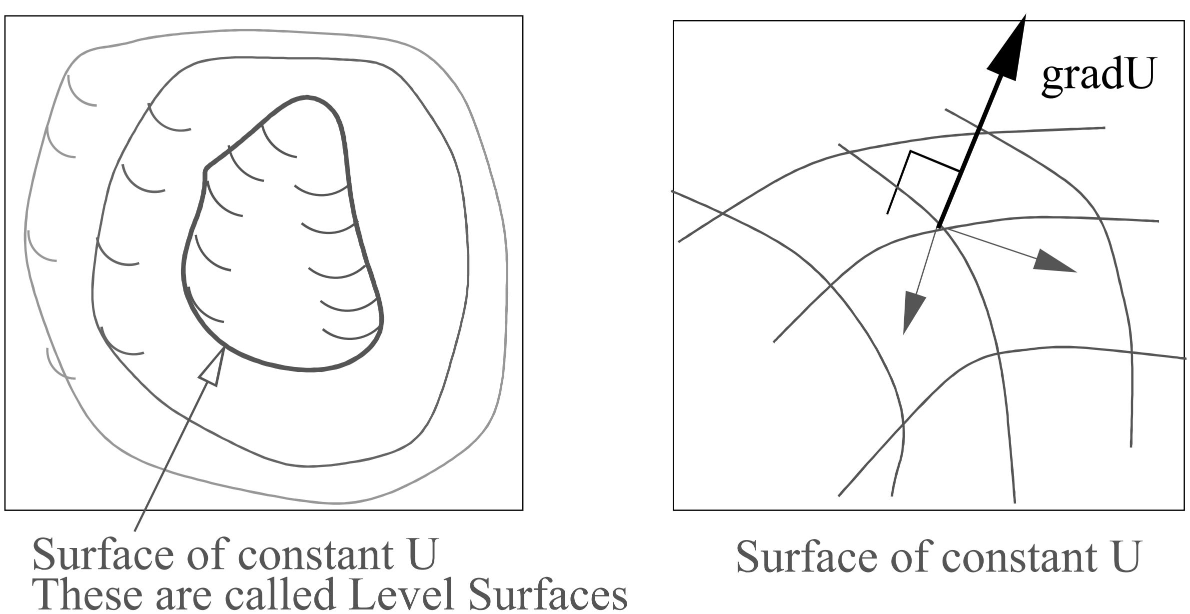

Figure 3:

The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

More precisely, if $\mathbf{a}(x, y, z)$ is a vector function of position in 3 dimensions, that is $\mathbf{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$, then its divergence at any point is defined in Cartesian co-ordinates by

$$\text{div}\mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

We can write this in a simplified notation using a scalar product with the $\boldsymbol{\nabla}$ vector differential operator:

$$\text{div}\mathbf{a} = \left(\hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right) \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot \mathbf{a}$$

Notice that the divergence of a vector field is a scalar field.

♣ Worked examples of divergence evaluation

$\mathbf{a}$	div$\mathbf{a}$	
$x\hat{\boldsymbol{\imath}}$	1	
$\mathbf{r}(= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$	3	
$\frac{1}{r^3}\mathbf{r}$	0	
$r\mathbf{c}$	$(\mathbf{r} \cdot \mathbf{c})/r$	where $\mathbf{c}$ is constant

Let us show the third example.

The x component of $\mathbf{r}/r^3$ is $x.(x^2 + y^2 + z^2)^{-3/2}$, and we need to find $\partial/\partial x$ of it.

$$\frac{\partial}{\partial x}x.(x^2 + y^2 + z^2)^{-3/2} = 1.(x^2 + y^2 + z^2)^{-3/2} + x\frac{-3}{2}(x^2 + y^2 + z^2)^{-5/2}.2x = r^{-3}\left(1 - 3x^2r^{-2}\right)$$

Adding this to similar terms for y and z gives

$$r^{-3}\left(3 - 3(x^2 + y^2 + z^2)r^{-2}\right) = r^{-3}(3 - 3) = 0$$

The significance of div

Consider a typical vector field, water flow, and denote it by $\mathbf{a}(\mathbf{r})$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{a}$ per unit time.

Now take an infinitesimal volume element dV and figure out the balance of the flow of $\mathbf{a}$ in and out of dV.

To be specific, consider the volume element $dV = dxdydz$ in Cartesian co-ordinates, and think first about the face of area $dxdz$ perpendicular to the y y direction. (That is, the one with surface area $d\mathbf{S} = -dxdz\hat{\boldsymbol{\jmath}}$.)

The component of the vector $\mathbf{a}$ normal to this face is $\mathbf{a} \cdot \hat{\boldsymbol{\jmath}} = a_y$, and is pointing inwards, and so the its contribution to the OUTWARD flux from this surface is

$$\mathbf{a} \cdot d\mathbf{S} \;=\; -\,a_y(y)dzdx.$$

(Flux here means mass per unit time.)

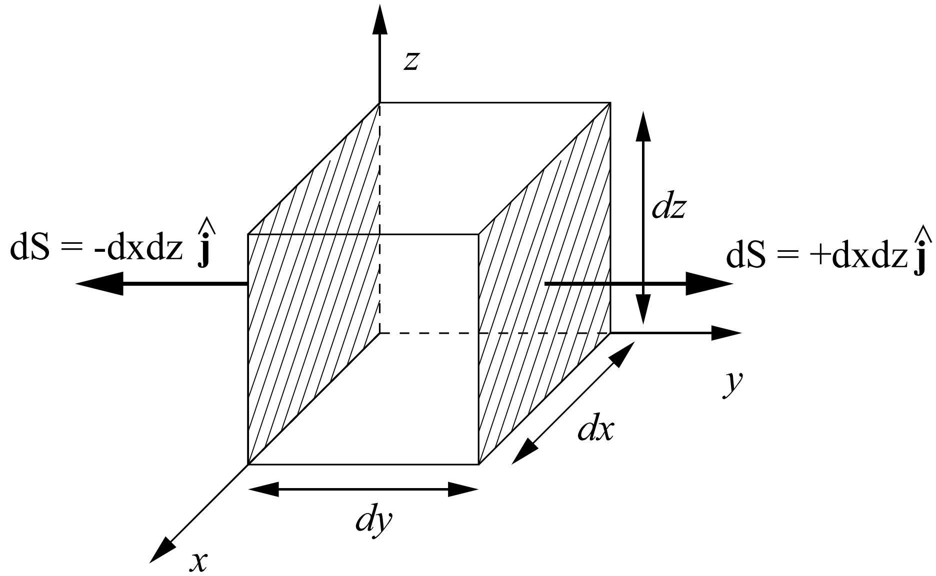

Figure 4: Elemental volume for calculating divergence.

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along y by an amount dy, so that this OUTWARD amount is

$$a_y(y+dy)dzdx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dxdz$$

The total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y}dydxdz = \frac{\partial a_y}{\partial y}dV$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) dV = \boldsymbol{\nabla}\cdot\mathbf{a}\ dV$$

Take home message: The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

(NB: The above does not constitute a rigorous proof of the assertion because we have not proved that the quantity calculated is independent of the co-ordinate system used, but it will suffice for our purposes.

The Laplacian: $\mathrm{div}(\mathrm{grad}U)$ of a scalar field

Recall that grad$U$ of *any* scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute div(grad$U$), even if we don't know what it means yet.

Here is where the $\boldsymbol{\nabla}$ operator starts to be really handy.

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}U) &= \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y}+\hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y}+\hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)U\right)\\
&= \left(\left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y}+\hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y}+\hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\right)U\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U\\
&= \left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)
\end{aligned}$$

This last expression occurs frequently in engineering science (you will meet it next in solving Laplace's Equation in partial differential equations). For this reason, the operator ∇^2 is called the "Laplacian"

$$\nabla^2 U=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U$$

Laplace's equation itself is

$$\nabla^2 U=0$$

♣ Examples of $\nabla^2 U$ evaluation

U	$\nabla^2 U$
$r^2(=x^2+y^2+z^2)$	6
xy^2z^3	$2xz^3+6xy^2z$
$1/r$	0

Let's prove the last example (which is particularly significant – can you guess why?).

$1/r=(x^2+y^2+z^2)^{-1/2}$ and so

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} &= \frac{\partial}{\partial x}-x.(x^2+y^2+z^2)^{-3/2}\\
&= -(x^2+y^2+z^2)^{-3/2}+3x.x.(x^2+y^2+z^2)^{-5/2}\\
&= (1/r^3)(-1+3x^2/r^2)
\end{aligned}$$

Adding up similar terms for y and z

$$\nabla^2\frac{1}{r}=\frac{1}{r^3}\left(-3+3\frac{(x^2+y^2+x^2)}{r^2}\right)=0$$

The curl of a vector field

So far we have seen the operator ∇

- Applied to a scalar field ∇U; and
- Dotted with a vector field $\nabla \cdot \mathbf{a}$.

You are now overwhelmed by that irrestible temptation to

- cross it with a vector field $\nabla \times \mathbf{a}$

This gives the **curl of a vector field**

$$\nabla \times \mathbf{a} \equiv \mathrm{curl}(\mathbf{a})$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\begin{aligned} \nabla \times \mathbf{a} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \\ &= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial y} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \hat{\boldsymbol{k}} \end{aligned}$$

♣ Examples of curl evaluation

$\mathbf{a}$	$\nabla \times \mathbf{a}$
$-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$	$2\hat{\boldsymbol{k}}$
$x^2y^2\hat{\boldsymbol{k}}$	$2x^2y\hat{\boldsymbol{\imath}} - 2xy^2\hat{\boldsymbol{\jmath}}$

The signficance of curl

Perhaps the first example gives a clue. The field $\mathbf{a} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ is sketched in Figure 5(a). (It is the field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega} = [0, 0, 1]$.) This field has a curl of $2\hat{\boldsymbol{k}}$, which is in the r-h screw out of the page. You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a} \cdot d\mathbf{r}$.

In fact curl is closely related to the line integral around a loop. The **circulation** of a vector $\mathbf{a}$ round any closed curve C is defined to be

$$\oint_C \mathbf{a} \cdot d\mathbf{r}$$

and the **curl** of the vector field $\mathbf{a}$ represents the **vorticity**, or **circulation per unit area**, of the field.

Our proof uses the small rectangular element dx by dy shown in Figure 5(b). Consider the circulation round the perimeter of a rectangular element.

Figure 5: (a) A rough sketch of the vector field $-y\hat{\imath} + x\hat{\jmath}$. (b) An element in which to calculate curl.

The fields in the x direction at the bottom and top are

$$a_x(y) \qquad \text{and} \qquad a_x(y+dy) = a_x(y) + \frac{\partial a_x}{\partial y} dy$$

and the fields in the y direction at the left and right are

$$a_y(x) \qquad \text{and} \qquad a_y(x+dx) = a_y(x) + \frac{\partial a_y}{\partial x} dx$$

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows, where the minus signs take account of the path being oppose to the field:

$$\begin{aligned} dC &= +[a_x(y)\,dx] + [a_y(x+dx)\,dy] - [a_x(y+dy)\,dx] - [a_y(x)\,dy] \\ &= +[a_x(y)\,dx] + \left[\left(a_y(x) + \frac{\partial a_y}{\partial x}dx\right) dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y}dy\right) dx\right] - [a_y(x)\,dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) dx\,dy \\ &= (\boldsymbol{\nabla} \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

where $d\mathbf{S} = dxdy\hat{\boldsymbol{k}}$.

NB: Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.

Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be **solenoidal**.
- A vector field with zero curl is said to be **irrotational**.
- A scalar field with zero gradient is said to be, er, well, **constant**.

Vector Operator Identities

In this lecture we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

We shall derive these using both conventional grunt, and using the compact notation. Note that the relevance of these identities may only become clear later in other Engineering courses.

Identity 1: curl grad U = $\nabla \times \nabla U = \mathbf{0}$

$$
\begin{aligned}
\nabla \times \nabla U &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\
&= \hat{\imath}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) + \hat{\jmath}\,() + \hat{k}\,() \\
&= \mathbf{0}\ .
\end{aligned}
$$

Note that $\nabla \times \nabla$ can be thought of as a null operator.

Identity 2: div curl a = $\nabla \cdot \nabla \times \mathbf{a} = 0$

$$
\begin{aligned}
\nabla \cdot \nabla \times \mathbf{a} &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\
&= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z}\frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \\
&= 0
\end{aligned}
$$

Identity 3: div and curl of $U\mathbf{a}$

Suppose that $U(\mathbf{r})$ is a scalar field and that $\mathbf{a}(\mathbf{r})$ is a vector field and we are interested in the product $U\mathbf{a}$, which is a vector field so we can compute its divergence and curl. For example the density $\rho(\mathbf{r})$ of a fluid is a scalar field, and the instantaneous velocity of the fluid $\mathbf{v}(\mathbf{r})$ is a vector field, and we are probably interested in mass flow rates for which we will be interested in $\rho(\mathbf{r})\mathbf{v}(\mathbf{r})$.

The divergence (a scalar) of the product $U\mathbf{a}$ is given by:

$$\begin{aligned}\nabla\cdot(U\mathbf{a}) &= \nabla\cdot(U\mathbf{a}) \\ &= U(\nabla\cdot\cdot\mathbf{a}) + (\nabla U)\cdot\mathbf{a} \\ &= U\mathrm{div}\mathbf{a} + (\mathrm{grad}U)\cdot\mathbf{a}\end{aligned}$$

In a similar way, we can take the curl of the vector field $U\mathbf{a}$, and the result should be a vector field:

$$\nabla\times(U\mathbf{a}) = U\nabla\times\mathbf{a} + (\nabla U)\times\mathbf{a}\ .$$

Identity 4: div of $\mathbf{a}\times\mathbf{b}$

Life quickly gets trickier when vector or scalar products are involved: For example, it is not *that* obvious that

$$\mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathrm{curl}\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathrm{curl}\mathbf{b}$$

To show this, use the determinant:

$$\begin{aligned}\begin{vmatrix}\partial/\partial x_i & \partial/\partial x_j & \partial/\partial x_k \\ a_x & a_y & a_z \\ b_x & b_y & b_z\end{vmatrix} &= \frac{\partial}{\partial x}[a_yb_z - a_zb_y] + \frac{\partial}{\partial y}[a_zb_x - a_xb_z] + \frac{\partial}{\partial z}[a_xb_y - a_yb_x] \\ &= \ldots \text{ bash out the products } \ldots \\ &= \mathrm{curl}\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot(\mathrm{curl}\ \mathbf{b})\end{aligned}$$

Vector operator identities in HLT

There is a kind of cottage industry in inventing vector identities. HLT contains a lot of them. So why not leave it at that?

First, since grad, div and curl describe key aspects of vectors fields, they arise often in practice, and so the identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.

Secondly, they help to identify other practically important vector operators. We now look at such an example.

Identity 5: $\text{curl}(\mathbf{a}\times\mathbf{b})$

$$\text{curl}(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix}$$

so the $\hat{\boldsymbol{\imath}}$ component is

$$\frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z)$$

which can be written as the sum of four terms:

$$a_x\left(\frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}\right] - b_x\left(\frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right] + \left(b_y\frac{\partial}{\partial y} + b_z\frac{\partial}{\partial z}\right)a_x - \left(a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)b_x$$

Adding $a_x\frac{\partial b_x}{\partial x}$ to the first of these, and subtracting from the last, and doing the same with $b_x\frac{\partial a_x}{\partial x}$ to the other two terms, we find that:

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = (\boldsymbol{\nabla}\cdot\mathbf{b})\mathbf{a} - (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{b} + [\mathbf{b}\cdot\boldsymbol{\nabla}]\mathbf{a} - [\mathbf{a}\cdot\boldsymbol{\nabla}]\mathbf{b}$$

where $[\mathbf{a}\cdot\boldsymbol{\nabla}]$ can be regarded as new, and very useful, scalar differential operator.

Definition of the operator $[\mathbf{a}\cdot\boldsymbol{\nabla}]$

This is a *scalar operator*, but it can obviously can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field:

$$[\mathbf{a}\cdot\boldsymbol{\nabla}] \equiv \left[a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right] \quad .$$

Identity 6: $\text{curl}(\text{curl}\mathbf{a})$ for you to derive

The following important identity is stated, and let as an exercise:

$$\text{curl}(\text{curl}\mathbf{a}) = \text{graddiv}\mathbf{a} - \nabla^2\mathbf{a}$$

where

$$\nabla^2\mathbf{a} = \nabla^2 a_x\hat{\boldsymbol{\imath}} + \nabla^2 a_y\hat{\boldsymbol{\jmath}} + \nabla^2 a_z\hat{\boldsymbol{k}}$$

♣ Example of Identity 6: electromagnetic waves

Background: Maxwell established a set of four vector equations which are fundamental to working out how eletromagnetic waves propagate. The entire telecommunications industry is built on these!

$$\begin{aligned} \text{div}\mathbf{D} &= \rho \\ \text{div}\mathbf{B} &= 0 \\ \text{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\ \text{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \end{aligned}$$

In addition, we can assume the following
$\mathbf{B} = \mu_r\mu_0\mathbf{H}, \qquad \mathbf{J} = \sigma\mathbf{E}, \qquad \mathbf{D} = \epsilon_r\epsilon_0\mathbf{E},$
where all the scalars are constants.

Question: Show that in a material with no free charge, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the eletric field $\mathbf{E}$ must be a solution of the wave equation

$$\nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0(\partial^2\mathbf{E}/\partial t^2) \, .$$

Answer: The good news is that we can do this knowing little about EM waves!

$$\begin{aligned}
\text{div}\mathbf{D} &= \text{div}(\epsilon_r\epsilon_0\mathbf{E}) = \epsilon_r\epsilon_0\text{div}\mathbf{E} = \rho = 0 \\
\text{div}\mathbf{B} &= \text{div}(\mu_r\mu_0\mathbf{H}) = \mu_r\mu_0\text{div}\mathbf{H} = 0 \\
\text{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \\
\text{curl}\mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = \mathbf{0} + \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t)
\end{aligned}$$

But we know (or rather you worked out) that curlcurl$\mathbf{E}$ = graddiv$\mathbf{E} - \nabla^2\mathbf{E}$, so

$$\text{curlcurl}\mathbf{E} = \text{curl}\left[-\mu_r\mu_0(\partial\mathbf{H}/\partial t)\right] = \text{graddiv}\mathbf{E} - \nabla^2\mathbf{E}$$

so interchanging the order of partial differentation, and using div$\mathbf{E} = 0$:

$$\begin{aligned}
-\mu_r\mu_0\frac{\partial}{\partial t}[\text{curl}\mathbf{H}] &= -\nabla^2\mathbf{E} \\
-\mu_r\mu_0\frac{\partial}{\partial t}\left[\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right] &= -\nabla^2\mathbf{E} \\
\mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} &= \nabla^2\mathbf{E}
\end{aligned}$$

Grad, div, curl and ∇^2 in curvilinear co-ordinate systems

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system by making use of the h factors

We recall that the unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial\mathbf{r}}{\partial u}$$

where $\mathbf{r}$ is the radius vector, and

$$h_u = \left|\frac{\partial\mathbf{r}}{\partial u}\right|$$

and similar expressions apply for the other co-ordinate directions. Then

$$d\mathbf{r} = h_u\hat{\mathbf{u}}du + h_v\hat{\mathbf{v}}dv + h_w\hat{\mathbf{w}}dw \, .$$

Grad in curvilinear coordinates

Using the properties of the gradient of a scalar field obtained previously,

$$\nabla U \cdot d\mathbf{r} = dU = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

The only way this can be satisfied for independent du, dv, dw is when

$$\nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial U}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial U}{\partial w}\hat{\mathbf{w}}$$

Divergence in curvilinear coordinates

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear co-ordinates by making use of the flux property.

We consider an element of volume dV. Although earlier we derived this as the volume of a parallelopiped, and found that the

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order in small quantities) and

$$dV = h_u\, h_v\, h_w\;\; du\, dv\, dw \;\; .$$

However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of u, v and w in general.

So the net efflux from the two faces in the $\hat{\mathbf{v}}$ direction shown in Figure 6.1 is

$$\begin{aligned} &= \left[a_v + \frac{\partial a_v}{\partial v}dv\right]\left[h_u + \frac{\partial h_u}{\partial v}dv\right]\left[h_w + \frac{\partial h_w}{\partial v}dv\right] dudw - a_v h_u h_w dudw \\ &= \frac{\partial(a_v h_u h_w)}{\partial v} dudvdw \end{aligned}$$

which is easily shown by multiplying the first line out and dropping second order terms (i.e. $(dv)^2$).

By definition div is the net efflux per unit volume, so summing up the other faces:

$$\begin{aligned} \text{div}\mathbf{a}\, dV &= \left(\frac{\partial(a_u\, h_v\, h_w)}{\partial u} + \frac{\partial(a_v\, h_u\, h_w)}{\partial v} + \frac{\partial(a_w\, h_u\, h_v)}{\partial w}\right) dudvdw \\ \Rightarrow \text{div}\mathbf{a}\; h_u h_v h_w\; dudvdw &= \left(\frac{\partial(a_u\, h_v\, h_w)}{\partial u} + \frac{\partial(a_v\, h_u\, h_w)}{\partial v} + \frac{\partial(a_w\, h_u\, h_v)}{\partial w}\right) dudvdw \end{aligned}$$

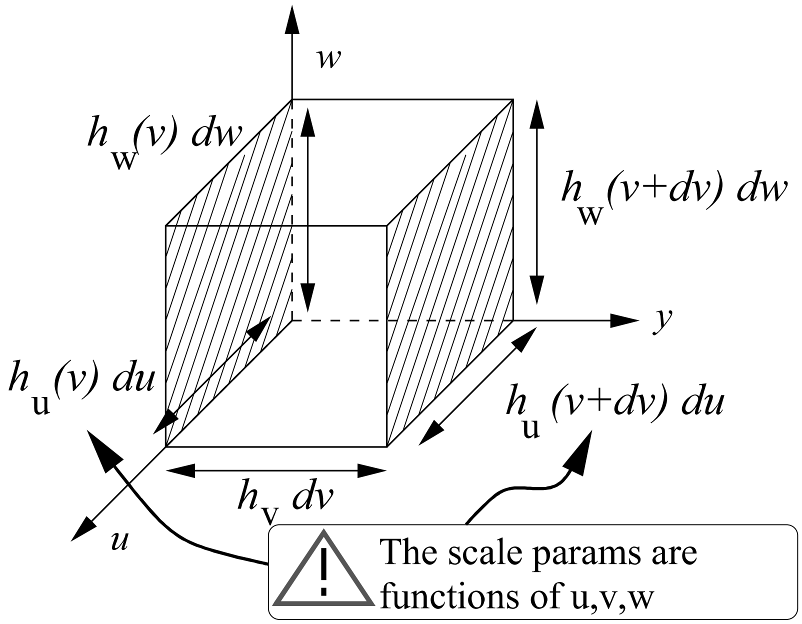

Figure 6.1: Elemental volume for calculating divergence in orthogonal curvilinear coordinates

So, finally,

$$\mathrm{div}\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u\, h_v\, h_w)}{\partial u} + \frac{\partial(a_v\, h_u\, h_w)}{\partial v} + \frac{\partial(a_w\, h_u\, h_v)}{\partial w}\right)$$

Curl in curvilinear coordinates

Recall from Lecture 5 that we computed the z component of curl as the circulation per unit area — the contents of the brackets () in the expression for the element of circulation

$$dC = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\, dx\, dy$$

By analogy with our derivation of divergence, you will realize that for an orthogonal curvilinear coordinate system we can write the area as $h_u h_v du dw$. But the opposite sides are no longer quite of the same length. The lower of the pair in Figure 6.2 is length $h_u(v)du$, but the upper is of length $h_u(v+dv)du$

Summing this pair gives a contribution to the circulation

$$a_u(v)h_u(v)du - a_u(v+dv)h_u(v+dv)du = -\frac{\partial(h_u a_u)}{\partial v}dvdu$$

and together with the other pair:

$$dC = \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right) dudv$$

So the circulation per unit area is

$$\frac{dC}{h_u h_v dudv} = \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)$$

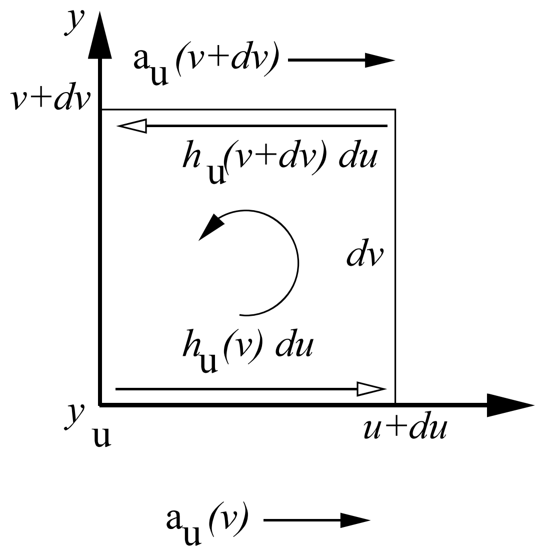

Figure 6.2: Elemental loop for calculating curl in orthogonal curvilinear coordinates

and hence curl is

$$\begin{aligned}\text{curl}\mathbf{a}(u,v,w) &= \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} + \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} + \\ &\quad \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}}\end{aligned}$$

The Laplacian in curvilinear coordinates

Substitution of the components of grad$U$ into the expression for div$\mathbf{a}$ immediately (!*?) gives the following expression for the Laplacian in general orthogonal co-ordinates:

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right] .$$

Grad, etc, in cylindrical polars

We recall that $\mathbf{r} = r\cos\phi\hat{\boldsymbol{\imath}} + r\sin\phi\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$, and that $h_1 = |\partial\mathbf{r}/\partial u|$, and so

$$\begin{aligned}h_1 &= \sqrt{(\cos^2\phi + sin^2\phi)} = 1, \\ h_2 &= \sqrt{(r^2\sin^2\phi + r^2\cos^2\phi)} = r, \\ h_3 &= 1\end{aligned}$$

Hence

$$
\begin{aligned}
\text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} \\
\text{div}\mathbf{a} &= \frac{1}{r}\left(\frac{\partial (r a_r)}{\partial r} + \frac{\partial a_\phi}{\partial \phi}\right) + \frac{\partial a_z}{\partial z} \\
\text{curl}\mathbf{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \frac{1}{r}\left(\frac{\partial (r a_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\hat{\boldsymbol{k}}
\end{aligned}
$$

The derivation of the expression for $\nabla^2 U$ in cylindrical polar co-ordinates is set as a tutorial exercise.

[redacted] Grad, etc, in spherical polars

We recall that $\mathbf{r} = r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}}$ so that

$$
\begin{aligned}
h_1 &= \sqrt{(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)} = 1 \\
h_2 &= \sqrt{(r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta)} = r \\
h_3 &= \sqrt{(r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi \\
\text{div}\mathbf{a} &= \frac{1}{r^2}\frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (a_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi} \\
\text{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(a_\phi\sin\theta) - \frac{\partial}{\partial \phi}(a_\theta)\right) + \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial \phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \\
&\quad \frac{\hat{\mathbf{e}}_\phi}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial \theta}(a_r)\right)
\end{aligned}
$$

- The derivation of the expression for $\nabla^2 U$ is set as a tutorial exercise.

♣ Examples

Q1 Find curl$\mathbf{a}$ in (i) Cartesians and (ii) Spherical poplars when $\mathbf{a} = x(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$.

A1 (i) In Cartesians, using the pseudo determinant gives curl$\mathbf{a} = -z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}$.

(ii) By inspection, in spherical polars

$$\mathbf{a} = r\sin\theta\cos\phi(r\hat{\mathbf{e}}_r)$$

so

$$a_r = r^2\sin\theta\cos\phi; \qquad a_\theta = 0; \qquad a_\phi = 0\,.$$

Hence

$$\begin{aligned}
\text{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(r^2\sin\theta\cos\phi)\right) + \\
&\quad \frac{\hat{\mathbf{e}}_\phi}{r}\left(-\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)\right) \\
&= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}(-r^2\sin\theta\sin\phi) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(-r^2\cos\theta\cos\phi)\right) \\
&= \hat{\mathbf{e}}_\theta(-r\sin\phi) + \hat{\mathbf{e}}_\phi(-r\cos\theta\cos\phi)
\end{aligned}$$

Now, these two results should be the same, but to check we need expressions for $\hat{\mathbf{e}}_r$ etc in terms of $\hat{\boldsymbol{\imath}}$ etc.

Remember that we can work out the unit vectors $\hat{\mathbf{e}}_r$ and so on in terms of $\hat{\boldsymbol{\imath}}$ etc using

$$\hat{\mathbf{e}}_r = \frac{1}{h_1}\frac{\partial\mathbf{r}}{dr}; \qquad \hat{\mathbf{e}}_\theta = \frac{1}{h_2}\frac{\partial\mathbf{r}}{d\theta}; \qquad \hat{\mathbf{e}}_\phi = \frac{1}{h_3}\frac{\partial\mathbf{r}}{d\phi} \quad \text{where } \mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\ .$$

Grinding through we find

$$\begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} = [R] \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix}$$

Don't be shocked to see a rotation matrix $[R]$: we are after all rotating one right-handed orthogonal coord system into another.

So the result in spherical polars is

$$\begin{aligned}
\text{curl}\mathbf{a} &= (\cos\theta\cos\phi\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{k}})(-r\sin\phi) + (-\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}})(-r\cos\theta\cos\phi) \\
&= -r\cos\theta\hat{\boldsymbol{\jmath}} + r\sin\theta\sin\phi\hat{\boldsymbol{k}} \\
&= -z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}
\end{aligned}$$

which is exactly the result in Cartesians.

Q2 Find the divergence of the vector field $\mathbf{a} = r\mathbf{c}$ where $\mathbf{c}$ is a constant vector (i) using Cartesian coordinates and (ii) using Spherical Polar coordinates.

A2 (i) Using Cartesian coords:

$$
\begin{aligned}
\text{div}\mathbf{a} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}c_x + \ldots \\
&= x.(x^2+y^2+z^2)^{-1/2}c_x + \ldots \\
&= \frac{1}{r}\mathbf{r}\cdot\mathbf{c} \ .
\end{aligned}
$$

(ii) Using Spherical polars

$$\mathbf{a} = a_r\hat{\mathbf{e}}_r + a_\theta\hat{\mathbf{e}}_\theta + a_\phi\hat{\mathbf{e}}_\phi$$

and our first task is to find a_r and so on. We can't do this by inspection, and finding their values requires more work than you might think! Recall

$$
\begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} = [R] \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix}
$$

Now the point is the same point in space whatever the coordinate system, so

$$a_r\hat{\mathbf{e}}_r + a_\theta\hat{\mathbf{e}}_\theta + a_\phi\hat{\mathbf{e}}_\phi = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$$

and using the inner product

$$
\begin{aligned}
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} \\
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top [R]^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} &= [R] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}
\end{aligned}
$$

For our particular problem, $a_x = rc_x$, etc, where c_x is a constant, so now we can write down

$$
\begin{aligned}
a_r &= r(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z) \\
a_\theta &= r(\cos\theta\cos\phi c_x + \cos\theta\sin\phi c_y - \sin\theta c_z) \\
a_\phi &= r(-\sin\phi c_x + \cos\phi c_y)
\end{aligned}
$$

Now all we need to do is to bash out

$$\text{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

In glorious detail this is

$$\begin{aligned}\text{div}\mathbf{a} &= 3\left(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z\right) + \\ &\qquad \frac{1}{\sin\theta}\Big((\cos^2\theta - \sin^2\theta)(\cos\phi c_x + \sin\phi c_y) - 2\sin\theta\cos\theta c_z\Big) + \\ &\qquad \frac{1}{\sin\theta}\left(-\cos\phi c_x - \sin\phi c_y\right)\end{aligned}$$

A bit more bashing and you'll find

$$\begin{aligned}\text{div}\mathbf{a} &= \sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z \\ &= \hat{\mathbf{e}}_r \cdot \mathbf{c}\end{aligned}$$

This is EXACTLY what you worked out before of course.

Take home messages from these examples:

All the trig was a bit tedious, but there are three important points!

- Just as physical vectors are independent of their coordinate systems, so are differential operators.
- Don't forget about the vector geometry you did in the 1st year. Rotation matrices are useful!
- Spherical polars were NOT a good coordinate system in which to think about this problem. Let the symmetry guide you.