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Study material for 1st semester general prepared by Biswajit Paul

1 Introduction to complex number:

No one person "invented" complex numbers, but controversies surrounding the use of these numbers existed in the sixteenth century. In their quest to solve polynomial equations by formulas involving radicals, early dabblers in mathematics were forced to admit that there were other kinds of numbers besides positive integers.Equations such as $x^2+2x+2=0$ and $x^3=6x+4$ that yielded "solutions" $1+\sqrt{-1}$ and $\sqrt[3]{2+\sqrt{-2}}+\sqrt[3]{2-\sqrt{-2}}$ caused particular consternation within the community of fledgling mathematical scholars because everyone knew that there are no numbers such as $\sqrt{-1}$ and $\sqrt{-1}$, numbers whose square is negative.Such "numbers" exist only in one's imagination, or as one philosopher opined, "the imaginary, (the) bosom child of complex mysticism." Over time these "imaginary numbers"did not go away, mainly because mathematicians as a group are tenacious and some are even practical.A famous mathematician held that even though "they exist in our imagination .... nothing prevents us from ... employing them in calculations." Mathematicians also hate to throw anything away.After all, a memory still lingered that negative numbers at first were branded "fictitious." The concept of number evolved over centuries; gradually the set of numbers grew from just positive integers to include rational numbers, negative numbers, and irrational numbers.But in the eighteenth century the number concept took a gigantic evolutionary step forward when the German mathematician Carl Friedrich Gauss put the so-called imaginary numbers or complex numbers, as they were now beginning to be called on a logical and consistent footing by treating them as an extension of the real number system.

Definition 1.1. A complex number is any number of the form $z=a+ib$ where a and b are real numbers and i is the imaginary unit.

1.1 Basic properties:

(i) The real number a in $z=a+ib$ is called the real part of z; the real number b is called the imaginary part of z. The real and imaginary parts of a complex number z are abbreviated $Re(z)$ and $Im(z)$, respectively. For example, if $z=3-2i$, then $Re(z)=3$ and $Im(z)=-2$.

(ii) A real constant multiple of the imaginary unit is called a pure imaginary number. For example,$z=6i$ is a pure imaginary number.

(iii) Two complex numbers are equal if their corresponding real and imaginary parts are equal.

(iv) The complex numbers can be visualized as the usual Euclidean plane by the following simple identification:
The complex number $z=x+iy\in\mathbb{C}$ is identified with the point $(x,y)\in\mathbb{R}^2$. For example, 0 corresponds to the origin and i corresponds to $(0,1)$. Naturally, the x and y axis of $\mathbb{R}^2$ are called the real axis and imaginary axis, because they correspond to the real and purely imaginary numbers, respectively (see figure1).

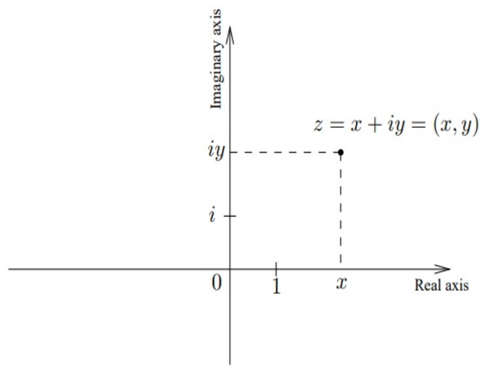

Figure 1: The complex plane

The natural rules for adding and multiplying complex numbers can be obtained simply by treating all numbers as if they were real, and keeping in mind that $i^2 = -1$. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + iy_1x_2)$$

If we take the two expressions above as the definitions of addition and multiplication, it is a simple matter to verify the following desirable properties:

(a) Commutative law: $z_1 + z_2 = z_2 + z_1, z_1z_2 = z_2z_1 \quad \forall z_1, z_2 \in \mathbb{C}$

(b) Associative law: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), (z_1z_2)z_3 = z_1(z_2z_3) \quad \forall z_1, z_2, z_3 \in \mathbb{C}$

(c) Additive inverse: For every complex number $z = (x, y)$, there is a unique $-z = (-x, -y) \in \mathbb{C}$ such that $z + (-z) = (-z) + z = 0$

(d) Multiplicative inverse: For every complex number $z \neq 0$, there is a unique $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = z^{-1}z = 1$

1.2 Exponential form of a complex number:

A nonzero complex number $z = r(cos\theta + isin\theta)$ can be written in the form $z = re^{i\theta}$ and it is called exponential form of complex number.

1.3 De Moivre's formula

Theorem 1. *If $z = cos(\theta) + isin(\theta)$ then $(cos(\theta) + isin(\theta))^n = cos(n\theta) + isin(n\theta), \forall \theta \in \mathbb{R}, n \in \mathbb{Z}$*

Application of De Moivre's formula :

(i) Compute $(1 + i)^{1000}$
Solution: The polar representation of $(1 + i)$ is $\sqrt{2}\left(cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right)$
Applying De Moivre's formula, we obtain $(1 + i)^{1000} = (\sqrt{2})^{1000}\left(cos\left(\frac{1000\pi}{4}\right) + isin\left(\frac{1000\pi}{4}\right)\right)$
$= 2^{500}\left(cos\left(250\pi\right) + isin\left(250\pi\right)\right) = 2^{500}$

(ii) Prove that

$$sin5t = 16sin^5t - 20sin^3t + 5sint$$
$$cos5t = 16cos^5t - 20cos^3t + 5cost$$

Solution: Using De Moivre's theorem to expand $(cost + isint)^5$, then using the binomial theorem, we have

$$\begin{aligned}cos5t{+}isin5t &= cos^5t + 5icos^4tsint + 10i^2cos^3tsin^2t + \\ &10i^3cos^2tsin^3t + 5i^4costsin^4t + i^5sin^5t \\ cos5t{+}isin5t &= cos^5t - 10cos^3(1 - cos^2t) + 5cost(1 - cos^2t)^2 + \\ &i(5(1 - sin^2t)^2sint - 10(1 - sin^2t)sin^3t + sin^5t)\end{aligned}$$

comparing real and imaginary part both sides and we get the desired result.

Definition 1.2. The function e^z defined by:

$$e^z = e^xcosy + ie^xsiny$$

is called the complex exponential function.

Some properties of exponential function: If z_1 and z_2 are complex numbers, then

(i) $e^0 = 1$

(ii) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(iii) $e^{z_1-z_2} = \dfrac{e^{z_1}}{e^{z_2}}$

(iv) $(e^{z_1})^n = e^{nz_1}, n = 0, \pm1, \pm2, \pm3, \pm4.......$

(v) $e^{z_1+2\pi i} = e^{z_1}$ (Periodicity)

Example: Find the values of the complex exponential function e^z at $z = 2 + \pi i$

Solution: For $z = 2 + \pi i$, we have $x = 2$ and $y = \pi$, and so $e^{2+\pi i} = e^2cos\pi + ie^2sin\pi$. Since $cos(\pi) = -1$ and $sin(\pi) = 0$, this simplifies to $e^{2+\pi i} = -e^2$.

1.4 Logarithm of a complex number

Definition 1.3. The multiple-valued function $Log(z)$ defined by:

$$Log(z) = log_e|z| + iarg(z) + i2n\pi, n = 0, \pm1, \pm2, \pm3......$$

is called complex logarithm. If $n = 0$ then it is called principal value of the logarithm and it is denoted by $log(z)$.

Example: Compute the complex logarithm $Log(z)$ for $z = 1 + i$.

Solution: It is clear that $|z| = \sqrt{2}$ and $arg(z) = \dfrac{\pi}{4}$.
Hence from the definition of $Log(z)$ we write,

$$\begin{aligned}Log(z) &= log_e\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right), n = 0, \pm1, \pm2, \pm3...... \\ &= \frac{1}{2}ln2 + i\left(2n\pi + \frac{\pi}{4}\right)\end{aligned}$$

The principal value is $log(z) = \frac{1}{2}log_e 2 + \frac{i\pi}{4}$

Remarks: For any non zero complex number a we define the power a^z as $a^z = e^{zLog(a)}$.

Example: Compute i^i
Solution: We write

$$\begin{aligned} i^i &= e^{iLog(i)} \\ &= e^{i\left(ln|i|+i\left(2n\pi+\frac{\pi}{2}\right)\right)} \\ &= e^{i\left(ln1+i\left(2n\pi+\frac{\pi}{2}\right)\right)} \\ &= e^{-\left(2n\pi+\frac{\pi}{2}\right)}, n = 0, \pm1, \pm2, \pm3...... \end{aligned}$$

1.5 Trigonometric and Hyperbolic Functions

Definition 1.4. The complex sine and cosine functions are defined by:

$$sinz = \frac{e^{iz} - e^{-iz}}{2i} \quad cosz = \frac{e^{iz} + e^{-iz}}{2}$$

similarly we define $tanz = \frac{sinz}{cosz}, cotz = \frac{cosz}{sinz}, cscz = \frac{1}{sinz}, secz = \frac{1}{cosz}$

Some properties of Trigonometric functions:

(a) $sin(-z) = -sin(z)$ and $cos(-z) = cos(z)$

(b) $cos^z + sin^2 z = 1$

(c) $sin(z_1 \pm z_2) = sin(z_1)cos(z_2) \pm cos(z_1)sin(z_2)$

(d) $cos(z_1 \pm z_2) = cos(z_1)cos(z_2) \mp sin(z_1)sin(z_2)$

Example: Find all solutions to the equation $sinz = 5$.

Solution: We use the definition above to solve the equation

$$\begin{aligned} & sinz = 5 \\ & \implies \frac{e^{iz} - e^{-iz}}{2i} = 5 \implies e^{2iz} - 10ie^{iz} - 1 = 0 \\ & \implies w^2 - 10iw - 1 = 0 \quad \text{let us put } w = e^{iz} \\ & \implies w = \frac{10i \pm \sqrt{-100+4}}{2} \implies w = (5 \pm 2\sqrt{6})i \\ & \implies e^{iz} = (5 \pm 2\sqrt{6})i \implies iz = Log((5 \pm 2\sqrt{6})i) \\ & \implies iz = ln(5 \pm 2\sqrt{6}) + i(2n\pi + \frac{\pi}{2}), \quad n = 0. \pm 1, \pm 2, ... \\ & \implies z = \left(2n\pi + \frac{\pi}{2}\right) - iln(5 \pm 2\sqrt{6}) \end{aligned}$$

Definition 1.5. The complex hyperbolic sine and hyperbolic cosine functions are defined by:

$$sinhz = \frac{e^z - e^{-z}}{2} \quad and \quad coshz = \frac{e^z + e^{-z}}{2}$$

similarly we define $tanhz = \dfrac{sinhz}{coshz}, cothz = \dfrac{coshz}{sinhz}, cschz = \dfrac{1}{sinhz}, sechz = \dfrac{1}{coshz}$

Some properties of Hyperbolic functions:

(a) $sinh(iz) = isin(z)$ and $cosh(iz) = cos(z)$

(b) $tan(iz) = itanhz$

(c) $sinh(-z) = -sinhz$ and $cosh(-z) = coshz$

(d) $cosh^z - sinh^2 z = 1$

(e) $sinh(z_1 \pm z_2) = sinh(z_1)cosh(z_2) \pm cosh(z_1)sinh(z_2)$

(f) $cosh(z_1 \pm z_2) = cosh(z_1)cosh(z_2) \pm sinh(z_1)sinh(z_2)$

1.6 Some selected problems:

1. Compute the following:
   (a) $i^{47} + i^{48} + i^{49} + ...i^{2021} + i^{2022}$
   (b) $i^{45}.i^{46}......i^{2021}.i^{2022}$
   (c) $\left(\dfrac{1+i}{1-i}\right)^{16} + \left(\dfrac{1-i}{1+i}\right)^{16}$
2. z_1, z_2 are complex numbers such that $z_1 + z_2$ and $z_1 z_2$ are both real. Prove that either z_1 and z_2 are purely real, or $z_1 = \bar{z}_2$.
3. Show that $|sinhz|^2 = sinh^2 x + sin^2 y$
4. Find the general value of $(1-4i)^{1+3i}$
5. find all general and principal values of $Log(-5), Log(-2-2i)$
6. Let$(1-\sqrt{3}i)^n = x_n + iy_n$, then show that
   (a) $x_n y_{n-1} - x_{n-1} y_n = 4^{n-1}\sqrt{3}$
   (b) Compute $x_n x_{n-1} - y_n y_{n-1}$
7. Compute $\dfrac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$

2 Theory of equation

Fundamental theorem of classical algebra:

Theorem 2. *every algebraic equation has a root, real or complex.*

Theorem 3. *An algebraic equation of degree n has n roots and no more.*

Proof. Let $f(x) = a_0 x^n + a_1 x^{n-l} + \ + a_n$ be a polynomial with coefficients real or complex, of degree n. Then $a_0 \neq 0$.
The equation $f(x) = 0$ is an algebraic equation of degree n. By the fundamental theorem, this equation has a root, say α_1.
Then $f(\alpha_1) = 0$ and by factor theorem, $(x - \alpha_1)$ is a factor polynomial $f(x)$.
Let $f(x) = (x - \alpha_1) f_1(x)$ where $f_1(x)$ is a polynomial of degree $n-1$ with leading coefficient a_0. By the

fundamental theorem, the equation $f_1(x) = 0$ has a root, α_2.
Then $f(\alpha_2) = 0$ and by factor theorem, $(x - \alpha_2)$ is a factor polynomial $f_1(x)$.
Let $f(x) = (x - \alpha_2)f_2(x)$ where $f_2(x)$ is a polynomial of degree $n - 2$ with leading coefficient a_0.
If $n > 2$, we continue with the same reasoning and come to some polynomial $f_{n-1}(x) = a_0(x - \alpha_n)$.

Therefore,

$$
\begin{aligned}
f(x) &= (x - \alpha_1)f_1(x) \\
&= (x - \alpha_1)(x - \alpha_2)f_2(x) \\
&\ldots\ldots\ldots\ldots \\
&= (x - \alpha_1)(x - \alpha_2)......(x - \alpha_{n-1})f_{n-1}(x) \\
&= a_0(x - \alpha_1)(x - \alpha_2)......(x - \alpha_{n-1})(x - \alpha_n)
\end{aligned}
$$

This shows that $f(x)$ is expressed as the product of n linear factors, each factor corresponds to a root and this proves that $\alpha_1, \alpha_2, ..., \alpha_n$ are n roots of the equation $f(x) = 0$.
Now we shall prove that there cannot be more than n roots. If possible, let β be a root of the equation $f(x) = 0$, where β is different from $\alpha_1, \alpha_2, ..., \alpha_n$. Since β is a root, $f(\beta) = 0$ and this would imply $a_0(\beta - \alpha_1)(\beta - \alpha_2)......(\beta - \alpha_{n-1})(\beta - \alpha_n) = 0$.
This is impossible because $a_0 \neq 0$ and $\beta - \alpha_i \neq 0$ for $i = 1, 2, .., n$. Therefore $f(x) = 0$ cannot have more than n roots.
This completes the proof. □

Theorem 4. *If an equation with real coefficients has a complex root $\alpha + i\beta$ then it has also the conjugate complex root $\alpha - i\beta$.*

Proof. Let $f(x) = 0$ be an equation of degree n with real coefficients and let $\alpha + i\beta$ be a root of $f(x) = 0$. It is obvious that $n \geq 2$.
Let us divide $f(x)$ by the product $\{x - (\alpha + i\beta)\}\{x - (\alpha - i\beta)\}$, i.e., by $(x - \alpha)^2 + \beta^2$. Let $q(x)$ be the quotient and $r(x)$ be the remainder.
Then the degree of $q(x)$ is $n - 2$ and the degree of $r(x)$ is at most one.
Since $f(x)$ is a real polynomial and $(x - \alpha)^2 + \beta^2$ is also a real quadratic, $q(x)$ and $r(x)$ are both real polynomials and we assume $r(x) = ax + b$ where a and b are both real.
Therefore $f(x) = [(x - \alpha)^2 + \beta^2]q(x) + ax + b$.
Since $\alpha + i\beta$ is a root, $f(\alpha + i\beta) = 0$, i.e., $a(\alpha + i\beta) + b = 0$
or, $(a\alpha + b) + ia\beta = 0$ and this implies $a\alpha + b = 0, a\beta = 0$.
But$\beta \neq 0$. Therefore $a = 0$ and consequently, $b = 0$.
So $f(x) = [(x - \alpha)^2 + \beta^2]q(x)$.
$f(\alpha - i\beta) = [(\alpha - i\beta - a)^2 + \beta^2]q(\alpha - i\beta) = 0$ and this proves that $(\alpha - i\beta)$ is a root of the equation $f(x) = 0$.
This completes the proof. □

Let us consider the following examples:
Examples:

1. Solve the equation $x^4 + x^2 - 2x + 6 = 0$, it given that $1 + i$ is a root.

   Solution: Let $f(x) = x^4 + x^2 - 2x + 6$.
   Since $f(x) = 0$ is an equation with real coefficients and $1 + i$ is a root of the equation, $1 - i$ is also a root.
   Therefore $(x - 1 - i)(x - 1 + i) = x^2 - 2x + 2$ is a factor of $f(x)$. Let $f(x) = (x^2 - 2x + 2)q(x)$. Then $q(x) = x^2 + 2x + 3$.
   $q(x) = 0$ gives $x = -1\sqrt{2}i$. Therefore the roots of the equation are $1 \pm i, -1\sqrt{2}i$.

2. Determine the multiple roots of the equation $x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2 = 0$

Solution: Let $f(x) = x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2$.
Then $f'(x) = 5x^4 + 8x^3 + 6x^2 + 8x + 1$.
The h.c.f of $f(x)$ and $f'(x)$ is $x^2 + 1 = (x + i)(x - i)$
Therefore the multiple roots of the equation are i and $-i$.

2.1 Descartes' rule of signs.

Statement of the rule:
The number of positive roots of an equation $f(x) = O$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $f(x)$ and if less, it is less by an even number.

Application of Descartes' rule of signs
Apply Descartes' rule of signs to examine the nature of the roots of the equation $x^4 + 2x^2 + 3x - 1 = 0$

Solution:
Let $f(x) = x^4 + 2x^2 + 3x - 1$.
Then $f(-x) = x^4 + 2x^2 - 3x - 1$.
The signs in the sequence of coefficients of $f(x)$ are $+ + + -$.
There is only one variation of signs and therefore the number of positive roots of$f(x) = O$ is exactly 1.
The signs in the sequence of coefficients of $f(-x)$ are $+ + - -$.
There is only one variation of signs and therefore the number of negative roots of $f(x) = 0$ is exactly 1.
The equation has no zero root. Therefore the number of real root is 2. The equation being of degree 4 has 4 roots. Consequently, the number of complex roots of the equation is 2.

2.2 Relation between roots and coefficients

Let $f(x) = f(x) = a_0x^n + a_1x^{n-l} + \ + a_n$ be a polynomial of degree n With coefficients real or complex. Then $a_0 \neq 0$.
Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of the equation $f(x) = 0$. Then

$$\begin{aligned}
&a_0x^n + a_1x^{n-1} + \ + a_n \\
=&a_0(x - \alpha_1)(x - \alpha_2)......(x - \alpha_{n-1})(x - \alpha_n) \\
=&a_0 \left[x^n - \sum \alpha_1 x^{n-1} + \sum \alpha_1\alpha_2 x^{n-2} - ......(-1)^n(\alpha_1\alpha_2....\alpha_n)\right],
\end{aligned}$$

where $\sum \alpha_1$ = sum of the roots,
$\sum \alpha_1\alpha_2$ = sum of the products of the roots taken two at a time,
... ...
$\sum \alpha_1\alpha_2.....\alpha_r$ = sum of the products of the roots taken r at a time.

From the equality of polynomials it follows that

$$a_1 = a_0\left(-\sum \alpha_1\right)$$
$$a_2 = a_0\left(\sum \alpha_1\alpha_2\right)$$
$$a_3 = a_0\left(-\sum \alpha_1\alpha_2\alpha_3\right)$$
$$......................$$
$$a_n = a_0(-1)^n\alpha_1\alpha_2.....\alpha_n$$

Therefore,

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$
$$\sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$$
$$\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$
$$..................$$
$$\alpha_1\alpha_2.....\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Example: Find the relation among p, q, r, s so that the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ is unity.

Solution: Let$\alpha, \beta, \gamma, \delta$ be the roots and $\alpha\beta = 1$. Then

$$\alpha + \beta + \gamma + \delta = -p \tag{1}$$
$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = q \tag{2}$$
$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r \tag{3}$$
$$\alpha\beta\gamma\delta = s \tag{4}$$
$$\text{From}(4), \gamma\delta = s \quad \text{and from}(3)\ (\gamma + \delta) + s(\alpha + \beta) = -r \tag{5}$$

From (1) & (5) $\alpha + \beta = \dfrac{r - p}{1 - s}$
From (1) $\gamma + \delta = -p - \dfrac{r - p}{1 - s} = \dfrac{ps - r}{1 - s}$
From (2) $\dfrac{r - p}{1 - s}\dfrac{ps - r}{1 - s} + 1 + s = q$
Or,$(r - p)(ps - r) = (1 - s)^2(q - s - 1)$.

2.3 Transformation of equations

When an equation is given it is possible, without knowing its individual roots, to obtain a new equation whose roots are connected With those of the equation by some assigned relation. The method of finding the new equation is said to be a transformation. Such a transformation sometimes helps us to study the nature of the roots of the given equation which would have been otherwise a difficult job.

Example: Transform the equation $x^4 + 4x^3 + 7x^2 + 6x - 4 = 0$ into one which shall want the second term and hence solve the given equation.

Solution: Let us apply the transformation $x = y + h$ so that the transformed equation may want the second term.

The transformed equation is

$$\begin{aligned}&(y+h)^4+4(y+h)^3+7(y+h)^2+6(y+h)-4=0\\&Or, y^4+y^3(4h+4)+y^2(6h^2+12h+7)+y(4h^3+12h^2+14h+6)+(h^4+4h^3+7h^2+6h-4)=0\end{aligned}$$

By the given condition $4h+4=0, i.e., h=-1$.
The equation reduces to $y^4+y^2-6=0$.
The roots of the transformed equation are $\pm\sqrt{2}, \pm\sqrt{3}i$.
Hence the roots of the given equation are $-1\pm\sqrt{2}, -1\pm\sqrt{3}i$.

2.4 Cardan's method:

Let the cubic equation be

$$ax^3+3bx^2+3cx+d=0 \tag{6}$$

This can be put in the standard form $z^3+3Hz+G=0$, where $z=ax+b, H=ac-b^2, G=a^2d-3abc+2b^3$.
To solve the equation, let us assume $z=u+v$.
Then $z^3=u^3+v^3+3uv(u+v)=u^3+v^3+3uvz \implies z^3-3uvz-(u^3+v^3)=0$.
Comparing this with $z^3+3Hz+G=0$, we have $uv=-H, u^3+v^3=-G$.
Therefore, $u^3=\dfrac{1}{2}(-G+\sqrt{G^2+4H^3}), v^3=\dfrac{1}{2}(-G-\sqrt{G^2+4H^3})$.

If p denotes any one of the three values of $\left\{\dfrac{1}{2}(-G+\sqrt{G^2+4H^3})\right\}^{\frac{1}{3}}$, then the three values of u are $p, \omega p, \omega^2 p$ where ω is an imaginary cube root of unity.
And since $uv=-H$, the three corresponding values of v are $\dfrac{-H}{p}, \dfrac{-\omega^2 H}{p}, \dfrac{-\omega H}{p}$.
Hence the values of z are $p-\dfrac{H}{p}, \omega p-\dfrac{\omega^2 H}{p}, \omega^2 p-\dfrac{\omega H}{p}$
and the three values of x are $\dfrac{1}{a}\left(p-\dfrac{H}{p}-b\right), \dfrac{1}{a}\left(\omega p-\dfrac{\omega^2 H}{p}-b\right), \dfrac{1}{a}\left(\omega^2 p-\dfrac{\omega H}{p}-b\right)$.
This gives the complete solution of the given equation.
The method of solution is called the Cardan's method of solution although the method owes its origin to Tartaglia.

Example: Solve the equation $x^3-18x-35=0$

Solution: Let $x=u+v$.
Then $x^3=u^3+v^3+3uvx$
or, $x^3-3uvx-(u^3+v^3)=0$
Comparing with the given cubic, we have $uv=6$ and $u^3+v^3=35$.
Therefore $u^3=\left\{\dfrac{1}{2}(35+\sqrt{35^2-864})\right\}=27$ and $v^3=\left\{\dfrac{1}{2}(35-\sqrt{35^2-864})\right\}=8$
The three values of u are $3, 3\omega, 3\omega^2$ and the three values of v are $2, 2\omega, 2\omega^2$. Since $uv=6$, we have $u+v=3+2, 3\omega+2\omega^2, 3\omega^2+\omega$.
Hence the roots of the given equation are $5, \dfrac{-5+\sqrt{3}i}{2}, \dfrac{-5-\sqrt{3}i}{2}$

2.5 Some exercise

1. Find the values of k for which the equation $x^4+4x^3-2x^2-12x+k=0$ has four real and unequal roots.

2. Solve the equations

    (a) $3x^3 - 26x^2 + 52x - 24 = 0$,

    (b) $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0$,

    (c) $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$,

    (d) $3x^4 + 20x^3 - 70x^2 - 60x + 27 = 0$,

    given that the roots are in geometric progression.

3. Solve the equation $2x^3 - 9x^2 + 7x + 6 = 0$ whose two roots α, β are connected by the relation $2\alpha + \beta = 1$.

4. The roots of the equation $x^3 + px^2 + qx + r = 0 (r \neq 0)$ are α, β, γ. Find the equation whose roots are $\frac{\alpha+\beta}{\gamma}, \frac{\beta+\gamma}{\alpha}, \frac{\gamma+\alpha}{\beta}$

5. Obtain the equation whose roots exceed the roots of the equation $x^4 + 3x^2 + 8x + 3 = 0$ by 1. Use Descartes' rule of signs to both the equations to find the exact number of real and complex roots of the given equation.

6. Apply Descartes' rule of signs to ascertain the minimum number of complex roots of the equation

    (a) $x^6 - 3x^2 - 2x - 3 = 0$

    (b) $x^7 - 3x^3 - x + 1 = 0$

    (c) $x^7 - 3x^3 + x^2 = 0$.

7. Solve by Cardan's method:

    (a) $x^3 - 12x + 8 = 0$

    (b) $x^3 - 27x - 54 = 0$

    (c) $x^3 - 9x + 28 = 0$

    (d) $x^3 + 9x^2 + 15x - 25 = 0$

    (e) $x^3 + 3x^2 - 3 = 0$

References

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